# Final-State Interactions in the Electrodisintegration of Deuterium*

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The cross section for the inelastic electron-deuteron scattering process is calculated using a semirelativistic approximation. The final-state interaction between the outgoing nucleons is estimated using approximate wave functions derived from the Gammel-Thaler potential. The rescattering correction is found to lead to a decrease in the peak value of the cross section, varying from about −5% at an electron momentum transfer of 1.4F$^{-1}$ to about −2% at a momentum transfer of 4F$^{-1}$. Various relativistic corrections are considered, and an ambiguity in the normalization of the semirelativistic wave functions is discussed. Finally, the neutron form factors are redetermined.

## 1. INTRODUCTION

In recent years considerable effort has been devoted to the study of the electromagnetic structure of the proton and the neutron. The inelastic electron-deuteron scattering process has been studied experimentally by the Stanford[1] and Cornell[2] groups, and theoretically by Blankenbecler,[3] Durand,[4] Goldberg,[5] Jankus,[6] and Bosco.[7] In principle, the neutron from factors may be determined from the differential cross section for this process, but there are considerable difficulties in the interpretation of the experimental results arising from relativistic effects and a possible final-state interaction of the two outgoing nucleons.

In the absence of a usable relativistic theory of the two-nucleon system rather drastic approximations must be made. Our calculation is similar to that of Durand.[4] The Hamiltonian is treated purely relativistically, and the two-nucleon system is described by approximate Breit wave functions, which effectively corresponds to treating the spins relativistically and the orbital part nonrelativistically.

The cross section has also been calculated numerically using Gammel-Thaler wave functions to estimate the final-state interaction.

In Sec. 2 the approximate wave functions are introduced, and in Sec. 3 the cross section is calculated in the absence of final-state interactions. These are estimated in Sec. 4, and the numerical work is described in Sec. 5. Finally, we discuss some of the approximations and other possible procedures for calculation in Sec. 6.

The following symbols will be used throughout: $K, K'$ = initial and final four-momentum of the electron; $p, p'$ = initial and final four-momentum of the proton; $P$ = initial four-momentum of the deuteron; $P'$ = final four-momentum of the center of mass of the outgoing two-nucleon system; $q=K-K'$ = momentum transferred by the electron; $k=\frac{1}{2}(p'-n')$ = relative momentum of the outgoing nucleons; $\theta$ = scattering angle of the electron.

All these quantities are measured in the laboratory system (the deuteron rest frame). The corresponding quantities in the center-of-mass frame of the outgoing nucleons will be denoted by the subscript $c$.

We also use $E$ for the energy of either nucleon in their center-of-mass frame, and $M$, $m$ for the masses of the nucleon and electron, respectively. The mass difference between the neutron and proton, and the deuteron binding energy are ignored. The electron will be treated as ultrarelativistic.

To avoid confusion, $q^2$ will be used only for the square of the four-momentum. The square of the three-momentum will be denoted by $|\mathbf{q}|^2$.

## 2. SEMIRELATIVISTIC CALCULATION OF THE CROSS SECTION

The process of interest is

$$e+d\rightarrow e+n+p.$$

The electromagnetic interaction may be treated in first Born approximation. (The validity of this has been discussed by Fubini *et al.*[8]) The deuteron and outgoing nucleons are described by Breit wave functions. Blankenbecler has shown[3] that if the retardation of the potential and the effect of nucleon-antinucleon pairs are ignored, then the energy shell matrix element is given by

$$T(\mathcal{E})=\langle f|j_\mu{}^e|i\rangle\int d^3t\,[\bar{\chi}(\mathbf{t},\mathbf{k}_c;P')j_\mu{}^p\phi(\mathbf{t}-\tfrac{1}{2}\mathbf{q};P)+\bar{\chi}(\mathbf{t},\mathbf{k}_c;P')j_\mu{}^n\phi(\mathbf{t}+\tfrac{1}{2}\mathbf{q},P)],\quad(2.1)$$

where $\bar{\chi}$ and $\phi$ are the momentum space Breit wave functions for the outgoing nucleons and the deuteron, respectively, and $j_\mu{}^e$, $j_\mu{}^p$, $j_\mu{}^n$ are the current operators for the electron, proton, and neutron.

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[1] C. de Vries, R. Herman, and R. Hofstadter, Phys. Rev. Letters 8, 381 (1962).
[2] D. N. Olson, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters 6, 286 (1961).
[3] R. Blankenbecler, Phys. Rev. 111, 1684 (1958).
[4] L. Durand, Phys. Rev. 123, 1393 (1961); 115, 1020 (1959).
[5] A. Goldberg, Phys. Rev. 112, 618 (1958).
[6] V. Z. Jankus, Phys. Rev. 102, 1586 (1956).
[7] B. Bosco, Nuovo Cimento 23, 1028 (1962).
[8] S. Fubini, M. Gourdin, and A. Martin, Nuovo Cimento 23, 249 (1962).

The full Breit wave functions are quite intractable. We shall assume they factorize into a product of free neutron and proton spinors and a scalar orbital wave function. To find a suitable form for the latter, it is useful to consider the ordinary Schrödinger wave function. In configuration space, the nonrelativistic (NR) deuteron wave function may be written

$$\phi_{\rm NR}(\mathbf{r})=\sum_{l}\frac{u_l(r)}{r}\langle 1m|lm_l1m_s\rangle Y_l^{m_l}(\hat{r})|1m_s\rangle_{\rm NR},\quad(2.2)$$

where $\langle 1m|lm_lsm_s\rangle$ is a Clebsch-Gordan coefficient defined as in reference 9 and $|1m_s\rangle_{\rm NR}$ is the nonrelativistic spin wave function. The normalization is given by

$$\int_0^\infty [u_0^2(r)+u_2^2(r)]dr=1.\quad(2.3)$$

The momentum space wave function is given by

$$\phi_{\rm NR}(\mathbf{s})=\frac{1}{(2\pi^2)^{1/2}}\sum_l \phi_l(|\mathbf{s}|)\langle 1m|lm_l1m_s\rangle Y_l^{m_l}(\hat{s})|1m_s\rangle,\quad(2.4)$$

where

$$\phi_l(|\mathbf{s}|)=\int_0^\infty \frac{u_l(r)}{r}j_l(|\mathbf{s}|r)r^2dr.\quad(2.5)$$

We assume that the orbital part of the Breit wave function is adequately approximated by the nonrelativistic form. Shirokov[10] has shown that in the deuteron rest frame the spin part of the relativistic wave function may be obtained from the nonrelativistic form merely by replacing the nonrelativistic spinors by relativistic spinors, providing the Foldy-Wouthuysen representation is used.[11] Hence, the Breit wave function is written in the form (2.2) with $|1m_s\rangle_{\rm NR}$ replaced by the relativistic spin function

$$|1m_s;\mathbf{s},P\rangle=\sum\langle 1m_s|\tfrac{1}{2}\mu_1\tfrac{1}{2}\mu_2\rangle v_p^{\mu_1}(\tfrac{1}{2}P+s)\times v_n^{\mu_2}(\tfrac{1}{2}P-s),\quad(2.6)$$

where the $v$'s are Foldy-Wouthuysen spinors for free particles, normalized by

$$\bar{v}v=1.$$

The final-state wave function is somewhat more difficult. Neglecting coupling between states of different angular momentum, the nonrelativistic configuration space wave function may be written as

$$\chi_{\rm NR}(\mathbf{r})=\sum_{JLSMM_LM_S}[4\pi(2L+1)]^{1/2}i^Le^{-i\delta_{JLS}}\times\left[\frac{F_{JLS}(|\mathbf{k}_c|r)}{|\mathbf{k}_c|r}\right]\langle JM|L0SM\rangle\times\langle JM|LM_LSM_S\rangle Y_L^{M_L}(\hat{r})|SM_S\rangle_{\rm NR},\quad(2.7)$$

where the normalization is chosen so that

$$\frac{F_{JLS}(|\mathbf{k}_c|r)}{|\mathbf{k}_c|r}\underset{r\to\infty}{\sim}\cos\delta_{JLS}\,j_L(|\mathbf{k}_c|r)+\sin\delta_{JLS}\,n_L(|\mathbf{k}_c|r).\quad(2.8)$$

This equation is written in the center-of-mass frame of the outgoing nucleons, whereas all the quantities in (2.1) are measured in the laboratory frame. The transformation between these frames has been discussed by MacFarlane[12] and Shirokov.[10] They show that the orbital part is essentially unchanged by the transformation apart from a normalization factor, and, in particular, its argument is still $\mathbf{k}_c$ rather than $\mathbf{k}$; while in the spin part we should use the laboratory values of the momenta. (It is also necessary to introduce various rotation matrices, but as these cancel out from the expression for the cross section in this case we shall omit them from the beginning.) Although their results do not apply strictly in the semirelativistic approach used here, the nature of their argument suggests it should be a very good approximation to use them in this case.

The choice of the normalization factor is a nontrivial problem. There are various possibilities for the normalization which are all equivalent in the strict nonrelativistic limit, but which differ appreciably in the region of interest. A formal expression for the normalization of the Breit wave function in the laboratory frame can be obtained by assuming that the potential is instantaneous in all frames, but this is likely to cause a large error for large values of the momentum transfers. Since it is known that using semirelativistic wave functions in the relativistic region is inconsistent, we require our normalization to give the best approximation to the relativistic result rather than to give consistency. For this reason the normalization has been fixed, rather arbitrarily by comparison with dispersion theory.

While the full dispersion calculation presents many difficulties, the contribution of the nucleon pole terms may be calculated straightforwardly.[13] The use of these terms only corresponds to neglecting final-state interactions and replacing the deuteron wave function by its asymptotic form. These approximations should be essentially unaffected by any relativistic effects, so the normalization has been fixed by requiring that in the neighborhood of the neutron pole the cross section should reduce essentially to the dispersion-theoretic result.

Finally, it is convenient to write the Breit wave function with an arbitrary axis. Then it has the form

$$\bar{\chi}(\mathbf{t},\mathbf{k}_c;P')=\sum\gamma[8(2L+1)]^{1/2}e^{+i\delta_{JLS}}\langle JM|L0SM\rangle\times\langle JM'|LM_LSM_S\rangle\mathfrak{D}_{M'M}{}^J(R)Y_L^{M_L*}(\hat{t}_c)\times\chi_{JLS}(\mathbf{t}_c,\mathbf{k}_c)\langle SM_S;\mathbf{t},P'|,\quad(2.9)$$

[9] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).
[10] M. J. Shirokov, Zk. Eksperim. i Teor. Fiz. **40**, 1387 (1961) [translation: Soviet Phys.—JETP **13**, 975 (1961)].
[11] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[12] A. MacFarlane, Rev. Mod. Phys. **34**, 41 (1962).
[13] S. Bose, Nuovo Cimento **17**, 767 (1960).

where

$$\chi_{JLS}(\mathbf{t}_c,\mathbf{k}_c)=\int_0^\infty dr\, r^2 j_L(|\mathbf{t}_c|r)\frac{F_{JLS}(|\mathbf{k}_c|r)}{|\mathbf{k}_c|r}, \quad (2.10)$$

$$\langle SM_S;\mathbf{t},P'|=\sum\langle SM_S|\tfrac{1}{2}m_1\tfrac{1}{2}m_2\rangle\bar{v}_p{}^{m_1}(\tfrac{1}{2}P'+t)\times\bar{v}_n{}^{m_2}(\tfrac{1}{2}P'-t), \quad (2.11)$$

and $\mathfrak{D}_{M'M}{}^J(R)$ is a rotation matrix representing the change of axis defined as in reference 9, where $R$ is the rotation that takes the axis into $\mathbf{k}_c$. $\gamma$ is a normalization factor, which has the value $(q_0+2M)/2E$ for the normalization discussed above. Physically, this is just the Lorentz contraction factor.

The only remaining factors in (2.1) are the currents. For a free particle, the proton current is given by[12]

$$\langle p'|j_\mu{}^p|p\rangle=e\bar{v}(p')\left[F_{1p}(-q^2)\gamma_\mu+\frac{i\kappa_p}{2M}F_{2p}(-q^2)\sigma_{\mu\nu}q_\nu\right]v(p) \quad (q=p'-p), \quad (2.12)$$

where $\kappa_p$ is the proton anomalous magnetic moment, and $F_{1p}$, $F_{2p}$ are real scalar functions normalized to

$$F_{1p}(0)=F_{2p}(0)=1.$$

We assume that this form is correct for the bound proton as well, and that $F_{1p}$, $F_{2p}$ have their free-particle values. This assumption is discussed by Blankenbecler.[3]

It is convenient to introduce linear combinations of $F_{1p}$ and $F_{2p}$ which represent the distribution of total charge and total magnetic moment[14]:

$$G_{\mathrm{CH}}{}^p(-q^2)=F_{1p}+\frac{\kappa_p q^2}{4M^2}F_{2p},$$
$$G_{\mathrm{M}}{}^p(-q^2)=\left(\frac{-q^2}{4M^2}\right)^{1/2}[F_{1p}+\kappa_p F_{2p}]. \quad (2.14)$$

These correspond to matrix elements of $j_\mu{}^p$ between states of definite helicity.[15] A similar form is assumed to hold for the neutron current. For the electron

$$\langle f|j_\mu{}^e|i\rangle=e\bar{v}_f\gamma_\mu v_i. \quad (2.15)$$

Finally, the differential cross section is given in terms of the energy-shell matrix element by

$$\frac{\partial^3\sigma}{\partial K_0'\partial\Omega(\mathbf{K}')\partial\Omega(\mathbf{k}_c)}=(2\pi)^{-5}m^2M^2\frac{K_0}{K_0'}\frac{|\mathbf{k}_c|}{2E}\frac{1}{6}\sum_{\text{spins}}|T(\mathcal{E})|^2. \quad (2.16)$$

### 3. FREE FINAL STATES

When there is no interaction between the outgoing nucleons, the final-state wave function has the simple form

$$\bar{\chi}(\mathbf{t},\mathbf{k}_c;P')=(2\pi)^3\delta(\mathbf{t}_c-\mathbf{k}_c)\gamma\bar{v}_p(\tfrac{1}{2}P'+t)\bar{v}_n(\tfrac{1}{2}P'-t). \quad (3.1)$$

Substituting this into the expression for the cross section and summing over the spins gives

$$|T(\mathcal{E})|^2=\frac{4\pi\gamma^2}{q^4}\frac{3e^4}{2m^2}\left[(G_{\mathrm{M}}{}^{p\,2}+G_{\mathrm{CH}}{}^{p\,2})\times\left\{\frac{[(p+p')\cdot(K+K')]^2}{(p+p')^2}+q^2\right\}-2q^2G_{\mathrm{M}}{}^{p\,2}\right]\times\left|\int d^3\mathbf{t}\,\delta(\mathbf{t}_c-\mathbf{k}_c)\phi_0(|\mathbf{k}-\tfrac{1}{2}\mathbf{q}|)\right|^2$$
$$+\text{neutron}+\text{interference terms}, \quad (3.2)$$

where the deuteron $D$ state has been ignored temporarily.

For any reasonable deuteron wave function, $\phi_0(|\mathbf{k}-\frac{1}{2}\mathbf{q}|)$ will be strongly peaked at $\mathbf{k}=\frac{1}{2}\mathbf{q}$. The laboratory and center-of-mass variables are connected by

$$\mathbf{k}-\tfrac{1}{2}\mathbf{q}=\mathbf{k}_c-\tfrac{1}{2}\mathbf{q}_c-\frac{[\mathbf{q}\cdot(\mathbf{k}-\frac{1}{2}\mathbf{q})]\mathbf{q}}{|\mathbf{q}|^2}\left[\frac{2E}{q_0+2M}-1\right]. \quad (3.3)$$

At

$$\mathbf{k}=\tfrac{1}{2}\mathbf{q},\quad \frac{2E}{q_0+2M}\approx1-\frac{|\mathbf{k}_c|^2}{2M^2}.$$

The second term in (3.3) is, therefore, much smaller than the first near $\mathbf{k}=\frac{1}{2}\mathbf{q}$, and $\mathbf{k}-\frac{1}{2}\mathbf{q}$ may be replaced by $\mathbf{k}_c-\frac{1}{2}\mathbf{q}_c$. This gives the maximum of the inelastic cross section at $\mathbf{k}_c=\frac{1}{2}\mathbf{q}_c=(M/2E)\mathbf{q}$, which agrees well with experiment.

Making this approximation in (3.2), a straightforward but tedious calculation gives

$$\frac{\partial^2\sigma}{\partial K_0'\partial\Omega(\mathbf{K}')}=\frac{\sigma_M}{\pi}M^2\frac{|\mathbf{k}_c|}{E}I_0(\theta), \quad (3.4)$$

where $\sigma_M$ is the Mott cross section for scattering by a point charge given by

$$\sigma_M=\frac{\alpha^2}{4K_0{}^2}\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}, \quad (3.5)$$

and

$$I_0(\theta)=\frac{4M^2}{4M^2-q^2}\left\{\left[\left[(G_{\mathrm{CH}}{}^p)^2+(G_{\mathrm{CH}}{}^n)^2+((G_{\mathrm{M}}{}^p)^2+(G_{\mathrm{M}}{}^n)^2)\left(1+\frac{4M^2-q^2}{2M^2}\tan^2\left(\frac{\theta}{2}\right)\right)\right]\times\left(M_0+\frac{|\mathbf{k}_c|^2}{2M^2}M_2\right)+\left[2G_{\mathrm{CH}}{}^pG_{\mathrm{CH}}{}^n+\tfrac{2}{3}G_{\mathrm{M}}{}^pG_{\mathrm{M}}{}^n\times\left(1+\frac{4M^2-q^2}{2M^2}\tan^2\left(\frac{\theta}{2}\right)\right)\right]\bar{M}_0\right\}+\Delta I, \quad (3.6)$$

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[14] F. J. Ernst, R. G. Sachs, and K. C. Wali, Phys. Rev. **119**, 1105 (1960).

[15] D. R. Yennie, D. G. Ravenhall, and M. M. Lévy, Rev. Mod. Phys. **29**, 144 (1957).

where

$$M_0=\tfrac{1}{2}\int_{-1}^{1}[\phi_0(z)]^2dz,$$

$$\bar{M}_0=\tfrac{1}{2}\int_{-1}^{1}\phi_0(z)\phi_0(-z)dz, \tag{3.7}$$

$$M_2=\tfrac{1}{2}\int_{-1}^{1}(1-z^2)[\phi_0(z)]^2dz,$$

and $\phi_0(z)$ is the function defined in (2.5) expressed in terms of $z=\cos\vartheta$, where $\vartheta$ is the angle between $\mathbf{k}_c$ and $\mathbf{q}_c$.

$\Delta I$ contains various negligibly small corrections to the interference terms, and also a term proportional to $q_{0c}M_1$, where

$$M_1=\tfrac{1}{2}\int_{-1}^{1}z[\phi_0(z)]^2dz. \tag{3.8}$$

At the peak in the inelastic cross section $q_{0c}\approx 0$ and this term is negligible. It gives an appreciable contribution, however, far away from the peak.

Noting that $4M^2-q^2=4E^2$ at the peak and substituting for $G_{\text{CH}}$ and $G_{\text{M}}$ from (2.14), this expression for the cross section agrees with that found by Durand[4] to within terms of order $q^2/M^2$ except for the term in $M_2$. This justifies the use of the four-momentum transfer in the cross section, but shows that appreciable deviations from the nonrelativistic result might be expected far from the peak, where the $\Delta I$ term becomes appreciable.

The $M_2$ term is a purely relativistic effect and corresponds to certain of the terms denoted by $\Delta_p$ in Appendix I of Durand's first paper.[4] It might be expected to be small, since $\phi_0$ is strongly peaked at $z=1$. A better estimate may be obtained by assuming a simple form for $u_0(r)$. If $u_0$ is taken to be a Hulthén wave function, the error in neglecting this term is of order $B/M$, where $B$ is the deuteron binding energy, which is of the same order as terms we have already dropped.

The effect of the deuteron $D$ state can be calculated in exactly the same way. The calculation is rather long and only the result will be given. The effect is to add a term

$$\frac{M^2}{E^2}\left\{\left[(G_{\text{CH}}{}^p)^2+(G_{\text{CH}}{}^n)^2+((G_{\text{M}}{}^p)^2+(G_{\text{M}}{}^n)^2)\times\left(1+\frac{2E^2}{M^2}\tan^2\left(\frac{\theta}{2}\right)\right)\right]N_0+\frac{4\sqrt{2}}{3}G_{\text{M}}{}^pG_{\text{M}}{}^n\left(1+\frac{2E^2}{M^2}\tan^2\left(\frac{\theta}{2}\right)\right)\bar{N}_0\right\} \tag{3.9}$$

to the peak value of $I_0$, where

$$N_0=\tfrac{1}{2}\int_{-1}^{1}\phi_0(z)\phi_2(z)dz,$$

$$\bar{N}_0=\tfrac{1}{2}\int_{-1}^{1}\phi_0(z)\phi_2(-z)dz,$$

and $\phi_2$ is given by (2.5).

In deriving (3.9) we have dropped contributions involving the square of the $D$-state wave function, and various terms of the nature of $M_2$.

## 4. EFFECT OF THE FINAL-STATE INTERACTION

Substituting the above expressions for the wave functions into the formula for $T(\mathcal{E})$, and neglecting the contributions from the deuteron $D$ state, gives a sum of Clebsch-Gordan coefficients, rotation matrices, and integrals of the form

$$\int d^3\mathbf{t}\left[\chi_{JLS}(\mathbf{t}_c,\mathbf{k}_c)Y_L{}^{*M_L}(\hat{t}_c)\phi_0(|\mathbf{t}-\tfrac{1}{2}\mathbf{q}|)e^{+i\delta_{JLS}}\times\langle SM_S;\mathbf{t},P'|j_\mu{}^p|1m,\mathbf{t}-\tfrac{1}{2}\mathbf{q},P\rangle\frac{\langle j_\mu{}^e\rangle}{q^2}\right]+\text{neutron}. \tag{4.1}$$

The final-state wave function should be strongly peaked at $\mathbf{t}_c=\mathbf{k}_c$. We may, therefore, replace $\mathbf{t}_c$ by $\mathbf{k}_c$ and $\mathbf{t}$ by $\mathbf{k}$ in the factors involving the currents; and shall also replace $\mathbf{t}-\frac{1}{2}\mathbf{q}$ by $\mathbf{t}_c-\frac{1}{2}\mathbf{q}_c$ in $\phi_0$ as before.

Writing

$$\phi_0(|\mathbf{t}_c-\tfrac{1}{2}\mathbf{q}_c|)=\textstyle\sum_l(2l+1)K_l(\mathbf{t}_c,\tfrac{1}{2}\mathbf{q}_c)P_l(\mathbf{t}_c,\tfrac{1}{2}\mathbf{q}_c), \tag{4.2}$$

where

$$K_l=\int_0^\infty j_l(|\mathbf{t}_c|r)j_l(\tfrac{1}{2}|\mathbf{q}_c|r)\frac{u_0(r)}{r}r^2dr, \tag{4.3}$$

the angular integration may be performed to give

$$T(\mathcal{E})=4\pi\textstyle\sum(2L+1)^{1/2}e^{i\delta_{JLS}}\langle JM|L0SM\rangle\times\langle JM'|LM_LSM_S\rangle\mathfrak{D}_{M'M}{}^J(R)\times\left[Y_L{}^{M_L*}(\hat{q})\frac{\langle j_\mu{}^e\rangle\langle j_\mu{}^p\rangle}{q^2}K_{JLS}(\mathbf{k}_c,\mathbf{q}_c)\right]+\text{neutron}, \tag{4.4}$$

where

$$K_{JLS}(\mathbf{k}_c,\mathbf{q}_c)=\frac{2}{\pi}\int_0^\infty t_c{}^2dt_c\int_0^\infty j_L(t_cr)\frac{F_{JLS}(|\mathbf{k}_c|r)}{|\mathbf{k}_c|r}r^2dr\times\int_0^\infty j_L(t_cr')j_L(\tfrac{1}{2}|\mathbf{q}_c|r')\frac{u_0(r')}{r'}r'^2dr'=\int_0^\infty\frac{F_{JLS}(|\mathbf{k}_c|r)}{|\mathbf{k}_c|r}j_L(\tfrac{1}{2}|\mathbf{q}_c|r)\frac{u_0(r)}{r}r^2dr. \tag{4.5}$$

As it stands, (4.4) is not the partial wave expansion of $T(\mathcal{E})$ since $\langle j_\mu{}^p\rangle$ depends on the orientation of $\mathbf{k}_c$, and the terms in the series are not orthogonal. In the neighborhood of the inelastic peak, $\mathbf{k}_c$ may be replaced by $\frac{1}{2}\mathbf{q}_c$ in $\langle j_\mu{}^p\rangle$. This is equivalent to dropping terms analogous to $M_1$ and $M_2$, and should, therefore, be well justified. A standard calculation then gives the cross section as an infinite series in $K_{JLS}^2$.

As pointed out by Durand,[4] this is not the best form for calculation, as the series converges rather slowly. It is better for calculation purposes to subtract the value of this expression when there is no final-state interaction and $K_{JLS}=K_L$, and to add this on again in closed form.

Introducing

$$\Delta_{JLS}=K_{JLS}^2-K_L^2 \tag{4.6}$$

the differential cross section is given by (3.4) with $I_0$ replaced by $I_0+I_{\mathrm{FS}}$ where, at the peak

$$I_{\mathrm{FS}}=\frac{M^2}{E^2}\Big\{\tfrac{1}{3}[a_1(G_{\mathrm{CH}}{}^p)^2+2a_2G_{\mathrm{CH}}{}^pG_{\mathrm{CH}}{}^n+a_1(G_{\mathrm{CH}}{}^n)^2]$$
$$+\tfrac{1}{6}[a_3(G_{\mathrm{M}}{}^p)^2+2a_4G_{\mathrm{M}}{}^pG_{\mathrm{M}}{}^n+a_3(G_{\mathrm{M}}{}^n)^2]$$
$$\times\left[1+\frac{2E^2}{M^2}\tan^2\left(\frac{\theta}{2}\right)\right]\Big\}, \tag{4.7}$$

with

$$a_1=\sum_{JL}(2J+1)\Delta_{J,L,1},$$
$$a_2=\sum_{JL}(-)^L(2J+1)\Delta_{J,L,1},$$
$$a_3=\sum_L[(3L+4)\Delta_{L+1,L,1}+(2L+1)\Delta_{L,L,1}+(3L-1)\Delta_{L-1,L,1}+2(2L+1)\Delta_{L,L,0}], \tag{4.8}$$
$$a_4=\sum_L(-)^L[(3L+4)\Delta_{L+1,L,1}+(2L+1)\Delta_{L,L,1}+(3L-1)\Delta_{L-1,L,1}-2(2L+1)\Delta_{L,L,0}].$$

### 5. NUMERICAL CALCULATIONS

For the numerical calculations $F_{JLS}$ was obtained as the solution of a Schrödinger equation with the Gammel-Thaler potential.[16] The potential was slightly modified by neglecting all terms coupling different partial waves. The deuteron was described by a Hulthén wave function with a hard core.[17] That is, with $x=\alpha r$, $x_c=\alpha r_c$

for $r\leq r_c$,

$$u_0(r)=u_2(r)=0,$$

and for $r>r_c$,

$$u_0(r)=A_0e^{-x}\{1-\exp[-\beta(x-x_c)]\},$$
$$u_2(r)=A_2e^{-x}\{1-\exp[-\gamma(x-x_c)]\}^2\times\left[1+\frac{3(1-e^{-\gamma x})}{x}+\frac{3(1-e^{-\gamma x})^2}{x^2}\right]. \tag{5.1}$$

It would have been preferable to use a Gammel-Thaler wave function for the deuteron as well, but this would have increased the computing time substantially, and was not worth while for the present experimental errors. The inclusion of the hard core in the Hulthén wave function should make it a better approximation, however.

[16] J. L. Gammel and R. M. Thaler, Phys. Rev. **103**, 1874 (1956).

[17] L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

TABLE I. Percentage contributions of various terms to the peak value of the cross section for typical values of $q^2$ and $\theta$.

| | $q^2=-4$ F$^{-2}$ $\theta=60°$ | $q^2=-4$ F$^{-2}$ $\theta=135°$ | $q^2=-10$ F$^{-2}$ $\theta=60°$ | $q^2=-10$ F$^{-2}$ $\theta=135°$ | $q^2=-16$ F$^{-2}$ $\theta=60°$ | $q^2=-16$ F$^{-2}$ $\theta=135°$ |
|---|---|---|---|---|---|---|
| S wave, $nn+pp$ | 102.1 | 102.4 | 100.1 | 99.8 | 100.8 | 100.2 |
| S wave, $np$ interference | 0.2 | −0.3 | −0.1 | 0.2 | −0.1 | 0.2 |
| D wave, $nn+pp$ | 1.1 | 1.1 | 1.1 | 1.2 | 1.3 | 1.3 |
| D wave, $np$ interference | −0.7 | −1.4 | −0.1 | −0.1 | 0.05 | 0.1 |
| Final-state Interaction | −2.7 | −1.8 | −1.6 | −1.0 | −2.0 | −1.7 |

The values of the parameters were taken to be $r_c=0.4$ F, $\alpha^{-1}=4.316$ F, $\beta=7.961$ F$^{-1}$, $\gamma=3.798$ F$^{-1}$, and $A_2/A_0=0.028$ corresponding to a 4% D-state probability and a deuteron effective range of 1.70 F. The results were not sensitive to the choice of parameters, and changing the D-state probability to 5% or the effective range to 1.73 F made a difference of less than 1% to the cross section. The presence of the hard core made a difference of 3 to 4% in the range considered. The percentage contribution of various terms to the differential cross section is given in Table I for typical values of $q^2$ and $\theta$.

The effect of the final-state interaction can be seen from Fig. 1, which shows the corrections $a_r$ as functions of $q^2$. The relative importance of different angular momentum states can be seen from Table II, which gives values of $\Delta_{JLS}$ for the first few partial waves. The corrections are negative throughout, in agreement with the approximate calculation by Durand[4] though the numerical values are somewhat larger. The corrections decrease with increasing momentum transfer and for $-q^2$ greater than 10 F$^{-2}$, they are negligible in comparison with the experimental error.

The Gammel-Thaler potential is not expected to be reliable for center-of-mass energies of the nucleons much greater than about 350 MeV which corresponds to $q^2=-16$ F$^{-2}$. This sets an upper limit to the range over which this calculation is useful. The numerical

TABLE II. Values of $D_{J,L,S}=\Delta_{J,L,S}/K_L^2$ for values of $q^2$ in F$^{-2}$.

| $-q^2$ | $L$ | $D_{L,L,0}$ | $D_{L,L,1}$ | $D_{L-1,L,1}$ | $D_{L+1,L,1}$ |
|---|---|---|---|---|---|
| 2 | 0 | −0.300 | ⋯ | ⋯ | −0.156 |
| | 1 | −0.182 | −0.060 | −0.072 | 0.066 |
| | 2 | 0.040 | 0.090 | −0.055 | −0.004 |
| | 3 | −0.013 | −0.000 | −0.000 | 0.000 |
| 6 | 0 | −0.235 | ⋯ | ⋯ | −0.217 |
| | 1 | −0.200 | −0.123 | −0.161 | 0.092 |
| | 2 | 0.079 | 0.103 | −0.113 | 0.013 |
| | 3 | −0.039 | −0.004 | −0.008 | 0.009 |
| 10 | 0 | −0.179 | ⋯ | ⋯ | −0.274 |
| | 1 | −0.206 | −0.162 | −0.219 | 0.040 |
| | 2 | 0.066 | 0.038 | −0.149 | 0.038 |
| | 3 | −0.045 | −0.007 | −0.018 | 0.020 |
| 16 | 0 | −0.176 | ⋯ | ⋯ | −0.382 |
| | 1 | −0.223 | −0.211 | −0.295 | −0.036 |
| | 2 | 0.023 | −0.036 | −0.193 | 0.064 |
| | 3 | −0.048 | −0.014 | −0.040 | 0.048 |

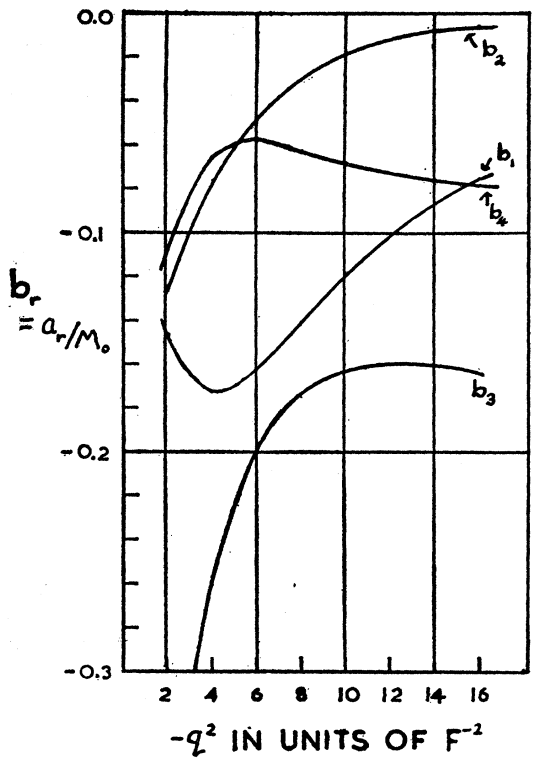


FIG. 1. Values of $b_r = -a_r/M_0$.

accuracy in the calculation of the corrections is not very great, but they should certainly be correct in order of magnitude and sign. It was difficult to estimate the sensitivity of the corrections to the choice of wave functions. These conclusions on the relative importance of the corrections are not affected by the choice of the wave function normalization.

The neutron form factors were obtained from values of the ratio of the proton-electron elastic-scattering cross section to the peak values of the inelastic deuteron-electron cross section at various scattering angles. The values of these ratios, and of the proton form factors were taken from deVries, Herman, and Hofstadter.[1] The cross sections had been corrected for radiative effects.[18]

For values of $-q^2$ between 4 and 16 $F^{-2}$, the equations for the form factors possessed a solution to within experimental error, and the solutions found by taking different combinations of scattering angles agreed fairly well. For $-q^2=2$ $F^{-2}$ the ellipses did not intersect, but in this region the corrections are large and are highly energy-dependent; so that a much more accurate calculation of the corrections may be necessary. In all cases we took the "right-hand" solution[1] for the neutron form factors. The values of $F_{1n}$ and $F_{2n}$ are given in Table III and are shown in Fig. 2; $G_{CH}{}^n$ and $G_M{}^n$ are shown in Fig. 3. In Fig. 2 the values of $F_{2p}$ are shown for comparison.

The errors in the numerical work were estimated to be less than 1%; and allowing for the various approximations made it is estimated that the theoretical cross sections should be correct to within 4 or 5%. The errors given for the form factors are only a rough estimate, and might be much larger if the errors on the cross sections are completely uncorrelated.

These estimates do not allow for the uncertainty in the normalization of the wave functions. This could cause an error of up to 15% in the value of the peak cross section at large momentum transfers. The normalization factor enters in such a way that it is impossible to determine it solely from electron deuteron scattering experiments, though it might be possible to estimate it from some other process such as deuteron photodisintegration. Fortunately, the form factors do not depend very sensitively on the normalization, and the maximum error on the neutron form factors due to the normalization is about 6% at the upper end of the range of $q^2$ considered, and correspondingly less at the lower end.

TABLE III. Values of the neutron form factors for $q^2$ in $F^{-2}$. The errors quoted are maximum errors.

| $-q^2$ | $F_{1n}$ | $F_{2n}$ |
|---|---|---|
| 4 | 0.110±0.20 | 0.750±0.10 |
| 6 | 0.130±0.10 | 0.662±0.08 |
| 8 | 0.112±0.16 | 0.604±0.06 |
| 10 | 0.100±0.10 | 0.500±0.05 |
| 12 | 0.085±0.11 | 0.435±0.06 |
| 14 | 0.062±0.08 | 0.376±0.05 |
| 16 | 0.075±0.06 | 0.338±0.03 |

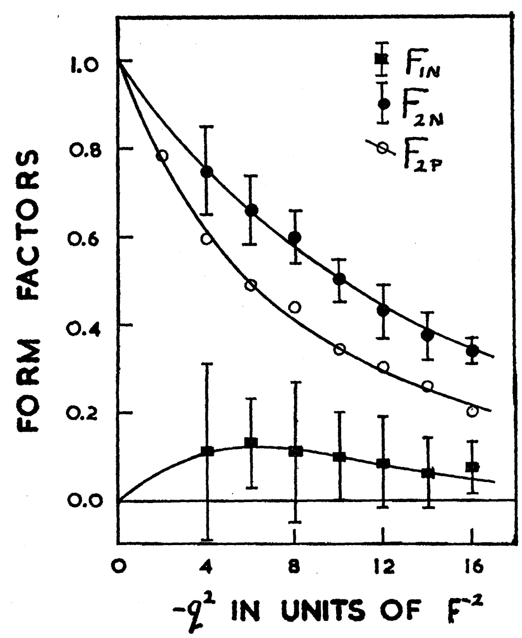


FIG. 2. Values of $F_{1n}$, $F_{2n}$, and $F_{2p}$. The errors shown are maximum errors, and the curves are smooth curves drawn through the experimental points.

[18] S. Sobottka, Phys. Rev. 118, 831 (1960).

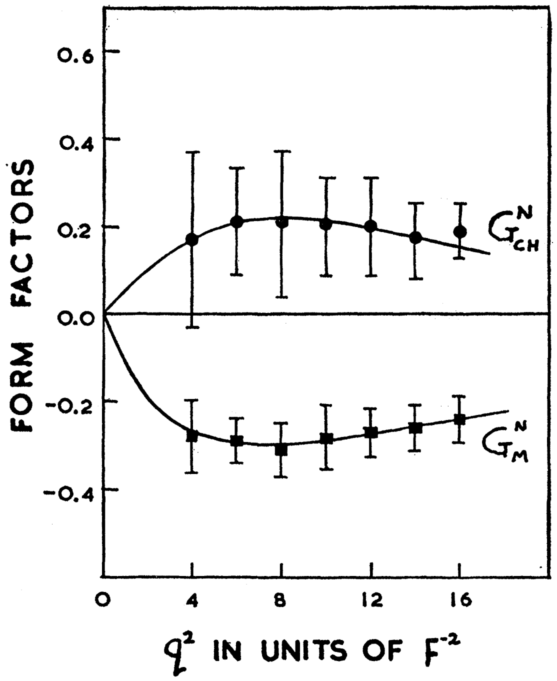


FIG. 3. Values of $G_{CH}{}^n$ and $G_M{}^n$. The errors shown are maximum errors, and the curves are smooth curves drawn through the experimental points.

The results for the form factors are compatible with $F_{1n}=0$, though a small positive value is suggested. There seems to be no indication of negative values for $F_{1n}$. The relationship $G_{CH}{}^n=0$, giving zero charge distribution for the neutron, does not seem to be supported, nor does, the relationship $G_M{}^n/K_n=G_M{}^p/(1+K_p)$, giving equal distributions of magnetic moment for the two nucleons. The hypothesis $F_{2n}=F_{2p}$ is almost certainly wrong. All these conclusions are essentially unaffected by the uncertainty in normalization.

## 6. DISCUSSION

In view of the somewhat arbitrary assumptions we have been forced to make, the question arises whether one might do better by using other techniques, notably dispersion relations. The difficulty in using dispersion techniques arises from the anomalous threshold due to the deuteron. While the pole approximation may be calculated straightforwardly, present techniques seem to be inadequate to cope with the effect of the rescattering correction, or any other terms in which the two nucleon amplitude occurs bilinearly.[19] The general form of the cross section can be deduced on general grounds—the difficulty arises in identifying the various terms that arise with the properties of the free neutron and proton, and in the absence of either a usable relativistic theory of a bound state or a systematic method of going off the mass shell, it is hard to see how this can be done without some such assumptions as we have made.

At the present time the experimental uncertainties are rather larger than the theoretical ones; the chief need at the moment is better experimental results, particularly at small momentum transfers and small scattering angles.

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[19] We should like to thank Professor R. E. Cutkosky for a discussion on this point.